AD-A049 771 YALE UNIV NEW HAVEN CONN SYSTEMS AND INFORMATION SCIENCES F/G 12/1
THE STABILITY OF SECOND ORDER QUADRATIC DIFFERENTIAL EQUATIONS,(U)
NOV 77 D E KODITSCHEK, K S NARENDRA N00014-67-C-0097-0020
UNCLASSIFIED SIS-7709 NL

| OF |

AD
A049771

END
DATE
FILMED
3-78
DDC

SYSTEMS AND INFORMATION SCIENCES

DEPARTMENT OF ENGINEERING
AND APPLIED SCIENCE
YALE UNIVERSITY
NEW HAVEN, CONNECTICUT

12

6

THE STABILITY OF SECOND ORDER QUADRATIC DIFFERENTIAL EQUATIONS

10 Daniel E. Koditschek [redacted] Kumpati S. Narendra

14 S / IS [redacted] -7709

Contract No. N00014-67-C-0097-0020 new

15

11 Nov[redacted]77

12 26p.

410215

B

The Stability of Second Order Quadratic Differential Equations

Daniel E. Koditschek and Kumpati S. Narendra

Yale University

1. Introduction:

Linear time-invariant differential equations have been studied over the past three decades and the stability properties of systems described by the vector differential equation $\dot{x} = Ax$ are known to be determined by the eigenvalues of the matrix A. In contrast, the wealth of techniques available for nonlinear systems - oscillations, asymptotic behavior, perturbation theory, etc. - reflects the obvious fact that a broad generalization of one theory is not applicable to all classes of systems. Even if our inquiries are limited to the two dimensional case, the volume of detailed work on special systems such as the Vander Pol or Duffings equations and the work of the Russian school in the early fifties on absolute stability demonstrates the need to treat these problems individually.

The results of this paper indicate that a profitable generalization of second order differential equations may be possible when we confine our attention to the quadratic case:

$$\dot{x} \triangleq \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x^T G x \\ x^T H x \end{bmatrix} \triangleq f(x) \qquad (1)$$

and its generalization

$$\dot{x} = Ax + f(x) \qquad (2)$$

which is considered in a subsequent paper. Moreover, the insights provided by this initial study will also aid in the analysis of higher order quadratic systems.

More than being a convenient class of nonlinear systems, quadratic differential equations have a traditional importance in stability literature. Given an arbitrary autonomous differential equation $\dot{x} = g(x)$, $g(x)$ may be expanded in a Taylor series if it satisfies certain regularity conditions. Lyapunov proved that if the equilibrium state of the linear approximation is asymptotically stable (unstable) the nonlinear system will also be asymptotically stable (unstable). However, in the critical case when the linear approximation is merely stable, the higher order terms must be examined to determine the nature of the stability of the equilibrium state and interest shifts to an equation of the form (2).

Recently the special class of bilinear systems

$$\dot{x} = Ax + u\,B\,x \tag{3}$$

has received a great deal of attention in the control literature and the principal results of this theory are influencing the direction of research on general nonlinear systems. When the control u(t) in equation (3) is a linear function of the state variables, equation (3) becomes a special case of (2) and the stability properties of such systems are bound to be of interest to control theorists. Further, quadratic differential equations are also known to arise in adaptive control [5] where the control parameters of a linear system are continuously adjusted and become state variables of a quadratic system.

In this paper ~~we undertake~~ a detailed study of the stability properties of
is undertaken.
the quadratic differential equation (A). After observing that such systems can
the quadratic differential equation
never be asymptotically stable the equilibrium states of (A) are classified in terms of the matrices G and H. Necessary and sufficient conditions for the stability of the origin are derived ~~in section~~ and constitute the principal contribution of this paper. Finally, these conditions are re-derived and elaborated using polar coordinates which allow a convenient classification of instability behavior. This exhausts the stability characteristics (in the sense of Lyapunov) of second order quadratic differential equations.

2. Homogeneous Systems of Even Degree:

Consider a dynamical system in $\mathbb{R}^n$ defined by

$$\dot{x} = h(x) \text{ where } h(cx) = c^k h(x) \text{ and } k \geq 1. \tag{4}$$

The "homogeneity" of h forces the direction of the field to be constant along any straight line through the origin. The consequences of this simple property pervade the following sections.

If $p(t;x_0)$ denotes the solution of (4) given initial condition $p(t_0;x_0) = x_0$, it can be easily shown [2] that $p(t;x_0) = \beta p(\beta^{k-1}t;x_0)$. Now suppose k is even: letting $\beta = -1$ we have

$$p(t;-x_0) = -p(-t;x_0) \tag{5}$$

which implies that any trajectory for $t \geq 0$ through x_0 has an associated trajectory through $-x_0$ for $t \leq 0$ which is its reflection. Hence, if Ω is a neighborhood of the origin and $\pm x_0 \notin \Omega$ but $\pm p(T;x_0) \in \Omega$ for some time $T > t_0$ then by (5) the trajectory $p(t;-p(T;x_0))$ will leave Ω after a finite time and pass through $-x_0$ after $T - t_0$ time has elapsed [Figure 1]. This, in turn, implies that an even homogeneous system can never be asymptotically stable.

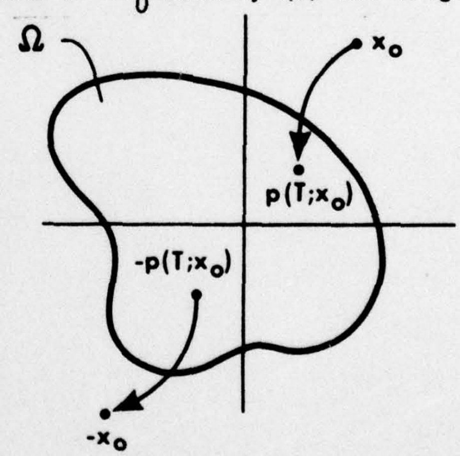


Figure 1. Reflection Property of Even Homogeneous System

With this general observation we set $k = n = 2$ and devote the remainder of the paper to the study of $\dot{x} = f(x)$ in $\mathbb{R}^2$.

Instability of an Isolated Singularity at the Origin:

The first direct consequence of the homogeneity property, (4), for the system (1) is that a second order homogeneous differential equation of second degree cannot be stable unless its field vanishes on an entire line.

Suppose the origin is stable: i.e. for any $\varepsilon > 0$ we can find a $\delta > 0$ such that trajectories starting in B_δ - the ball of radius δ around the origin - always remain in B_ε. From the preceding discussion this system cannot be attractive. Hence, for any $\mu > 0$ we can find a $\nu > 0$ such that trajectories which start outside B_μ never penetrate B_ν. Then by suitably choosing $\nu < \mu < \delta < \varepsilon$, any trajectory starting in $\Omega_1 \overset{\Delta}{=} B_\delta - B_\mu$ will remain in $\Omega_2 \overset{\Delta}{=} B_\varepsilon - B_\nu$ for all $t \geq 0$, and hence in $\bar{\Omega}_2$ (see fig. 2). By the Poincaré-Bendixson theory [3] any autonomous trajectory in a closed subset of R^2 must be either attracted to a singularity or must be a closed path. Any closed path trajectory in Ω_2 must cross the line through x_0 and the origin in at least two different directions. But this violates homogeneity, hence $\bar{\Omega}_2$ must contain a singular point. By construction, this cannot be the origin. Since $f(y) = 0$, $y \neq 0$ implies $f(\alpha y) = 0$ by homogeneity, the field must vanish on the entire line through this singular point and the origin.

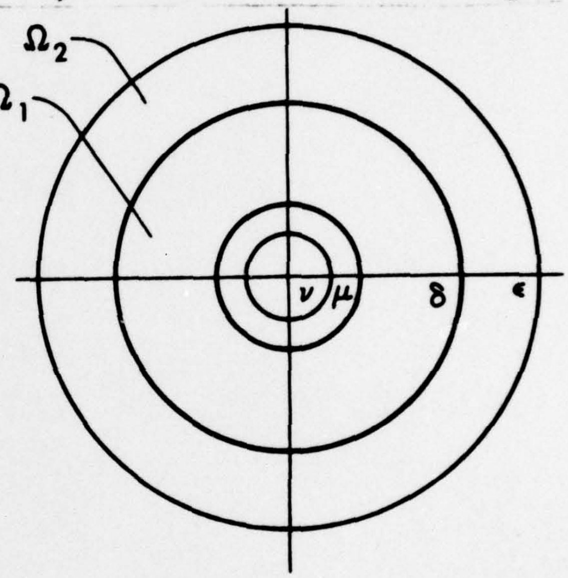


Figure 2.

The next section will investigate the existence of such lines for the quadratic differential equation.

3. Equilibria of Second Order Quadratic Systems:

We now consider the particular class of second degree second order systems described by (1).

$$\begin{aligned} \dot{x}_1 &= x^T G x \\ \dot{x}_2 &= x^T H x \end{aligned} \tag{1}$$

with which this paper is chiefly concerned. We will assume that neither G nor H is identically zero, and, without loss of generality, that both are symmetric. From the discussion in the previous section it is clear that the locus of the set of critical points of f is crucial to the stability properties of (1). Since this is completely determined by G and H, well-known properties of symmetric matrices yield the following exhaustive classification of the types of equilibrium states of the quadratic differential equation. The field in (1) may vanish

(i) only at the origin

(ii) along a straight line through the origin

(iii) along two straight lines through the origin.

If either G or H is definite, the system is obviously of type (i). If G and H are indefinite, but of full rank, the system may be of type (i), (ii), or (iii). If G is singular and H is indefinite and of full rank (or vice versa) the system is of type (i) unless H maps the zero eigenvector of G into its orthogonal complement (i.e. $Gx = 0$ & $x^T Hx = 0$) in which case the system is of type (ii). If G and H are both singular then the system is of type (i) unless $G = \alpha H$, in which case we get type (ii). These cases are illustrated in the following examples:

<u>Example 3.1</u>: G & H full rank, indefinite.

(a) If $G = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$ and $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, then $x^T Gx = x_1 x_2 = 0$ is satisfied by the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $x^T Hx = x_1^2 - x_2^2 = 0$ is satisfied by the vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, but $\begin{bmatrix} x^T Gx \\ x^T Hx \end{bmatrix} = 0$ has only the trivial solution. The system is type (i).

(b) If $G = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$ and $H = G^{-1} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = 4G$ then $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ both satisfy $\dot{x} = \begin{bmatrix} x^T Gx \\ x^T Hx \end{bmatrix} = \begin{bmatrix} x_1 x_2 \\ 4x_1 x_2 \end{bmatrix} = 0$ and the system is type (iii).

(c) If $G = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$ and $H = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 0 \end{bmatrix}$, then $x^T Gx = x_1 x_2 = 0$ for $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $x^T Hx = x_1(x_1 + x_2) = 0$ for $\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Hence $\dot{x} = 0$ along $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and the system is type (ii).

<u>Example 3.2</u>: G & H singular.

(a) If $G = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $H = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, then $\lambda \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ satisfies $x^T Gx = x_2^2 = 0$ uniquely, and $\lambda \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ satisfies $x^T Hx = x_1^2 = 0$ uniquely so the system is type (i).

(b) If $G = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $H = -G = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$ then $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ satisfies $\begin{bmatrix} x^T Gx \\ x^T Hx \end{bmatrix} = \begin{bmatrix} x_1^2 \\ -x_1^2 \end{bmatrix} = 0$ uniquely.

<u>Example 3.3</u>: G indefinite, full rank; H singular.

(a) If $G = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$ and $H = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ then $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ satisfy $x^T Gx = x_1 x_2 = 0$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ satisfies $x^T Hx = x_1^2 + 2x_1 x_2 + x_2^2 = 0$ uniquely, hence the origin is the only equilibrium.

(b) If $G = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$ and $H = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$ then $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, the zero eigenvector of H, is mapped by G into $\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$ - its orthogonal complement. The system is of type (ii).

In summary, the necessary condition of a zero line stipulates that stable systems must be of the kind 3.1.b, 3.1.c, 3.2.b, 3.3.b. Sufficient conditions are derived in the next section.

4. Stability of Second Order Quadratic Systems:

From the discussion in section 2 it follows directly that any quadratic system of type (i), with a unique isolated equilibrium at the origin, is unstable. In this section we shall further refine necessary conditions and finally arrive at conditions which are both necessary and sufficient for stability. This is most easily accomplished by first deriving a special form for $f(x)$ in (1) that characterizes systems of type (ii) and (iii).

Quadratic Differential Equations with Non-isolated Equilibria:

If the origin is not an isolated equilibrium (i.e. if the system is not of type (i)) then let $\|e\| = 1$ and $f(e) = 0$ (or $e^TGe = e^THe = 0$). By homogeneity (4) we know the whole line λe ($\lambda \in \mathbb{R}$) is a set of critical points of f. Now define a new coordinate system by the orthogonal transformation $y = R^Tx$ where $R \triangleq [e \vdots c]$ ($\|e\| = \|c\| = 1$ and $e^Tc = 0$). Then

$$f(x) = f(Ry) = \begin{bmatrix} y^TPy \\ \\ y^TQy \end{bmatrix}$$

where $P = \begin{bmatrix} 0 & e^TGc \\ c^TGe & c^TGc \end{bmatrix}$ and $Q = \begin{bmatrix} 0 & e^THc \\ c^THe & c^THc \end{bmatrix}$.

Hence we can factor a $y_2 = c^Tx$ out of both quadratic forms to write

$$f(x) = c^Tx\, Dx \tag{7}$$

x where $D = \begin{bmatrix} 2e^TGc & c^TGc \\ 2e^THc & c^THc \end{bmatrix} R^T = \begin{bmatrix} c^TG \\ \hline c^TH \end{bmatrix} R \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} R^T$

Equation (7) expresses the fact that any quadratic system whose field vanishes along at least one line has solutions which are associated with a linear system

$$\dot{z} = \beta Dz \quad (\beta \in \mathbb{R}) \tag{7ℓ}$$

Given a fixed initial condition, z_0, the solution of (7ℓ) for any constant β is $z_\beta(t) = e^{D\beta t}z_0$, thus $z_{\beta_1}(t) = z_{\beta_2}(\tau)$ where $\tau = \frac{\beta_1}{\beta_2}t$. In other words, the trajectories of (7ℓ) cut out identical manifolds in R^2 for each β, although their time parametrization on these manifolds is dilated or contracted by an appropriate constant. More generally, if we are given

$$\dot{x} = \beta(t)Dx \tag{7t}$$

then $x(t) = e^{D\int_0^t \beta(\tau)d\tau}x_0$. Again, the solution of (7t) for any scalar time function $\beta(t)$ given initial condition x_0 will be contained in the same manifold as $z_{\beta_1}(t)$; however, its time parametrization will be varied by the function $\int_0^t \beta(\tau)d\tau$. Thus, if we think of (7) as a particular case of (7t) where $\beta(t) = c^Tp(t;x_0)$ has been computed a priori, then we see that all trajectories of the quadratic differential equation with a non-isolated set of equilibria must lie on manifolds determined by the linear system (7ℓ). With this observation in mind, we may now extend necessary conditions for stability by considering different classes of linear second order time-invariant systems. Specifically we will examine the nature of (7) when the equilibrium state of the linear system (7ℓ) is a node, center or focus.

a. <u>D singular</u>:

Suppose $D = kb^T$ - a singular matrix. Then (7) may be re-written as

$$\dot{x} = \left[x^Tbc^Tx\right]k \tag{8}$$

The symmetric part of bc^T is indefinite when $b \neq c$. In this case (8) is of the form $\dot{x} = k\,x^TGx$, hence $G = k_2/k_1\,H$ is indefinite and from the discussion in section 3 this is the only possible occurrence of a system type (iii). Clearly this equation defines a field whose direction is uniformly specified by the vector k and which vanishes on the two lines λe_1, λe_2 ($\lambda \varepsilon \mathbb{R}$), satisfying $e_i^TGe_i = 0$. If $k^TGk \neq 0$

then either $p(t;k)$ or $p(t;-k)$ will tend to infinity along the vector k (or $-k$) as $t \to \infty$ since $x^T Gx \Big|_{x = \lambda(t)k}$ has the same sign for $t \geq t_0$. This is illustrated in figure 3.a.

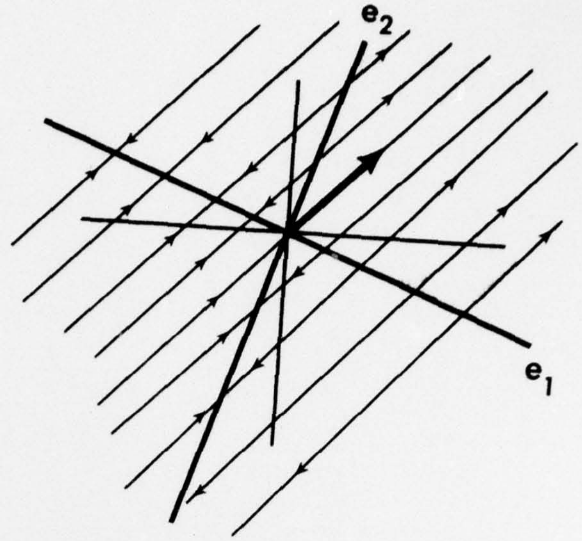


Figure 3.a. Type (iii): two zero lines unaligned with field direction.

A more interesting case arises when $x^T Gx = 0$ is satisfied on the line determined by k. Let r also satisfy this equality ($r \neq k$) and choose the four points $p = \pm k \pm \alpha r$ ($\alpha \in R$). The field vector at these points is either in the positive or negative direction of k as determined by $p^T Gp = 2\alpha k^T Gr$ or $-2\alpha k^T Gr$. If $\alpha > 0$ and we assume $k^T Gr < 0$ (note $k^T Gb \neq 0$ since $k \neq b$) it follows immediately that $p(t;k - \alpha r) \to \infty$ in the direction of k, and $p(t;-k - \alpha r) \to \infty$ in the direction of negative k as shown in figure 3.b. If $k^T Gr < 0$ then the alternate pair of trajectories is unstable. This exhausts the possible occurrences of type (iii) equations. Hence necessary conditions for stability have been extended to exclude all but systems of type (ii) - with a single line of equilibria.

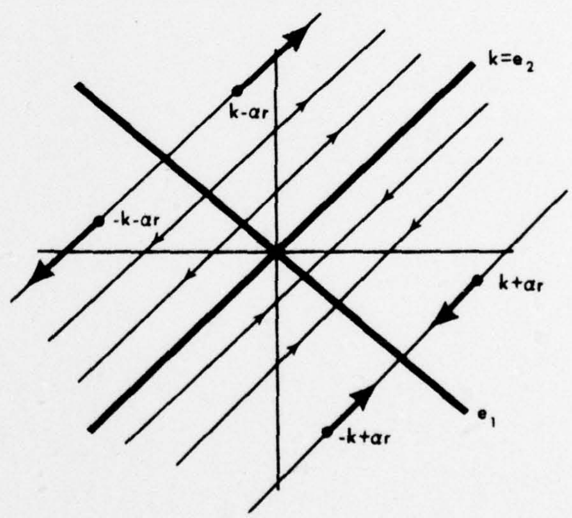


Figure 3.b. Type (iii): field direction aligned with one of two zero lines.

The remaining case of singular D occurs when the matrix bc^T in (8) is symmetric: i.e. $b = c$. If $k \neq c_\perp$ then either $p(t;k)$ or $p(t;-k)$ follows the line determined by k to infinity. If $k = c_\perp$ then equation (7) may be written $x = c_\perp x^T cc^T x$ and every trajectory which has non-zero motion tends to infinity in the direction $c_\perp$ as shown in figure 4.

b. D Non-Singular with Real Eigenvalues:

If the equilibrium of (7ℓ) is a node (i.e. the eigenvalues of D are real and non-zero) then its associated quadratic system (7) must be unstable. If D has an eigenvector, k, which is not orthogonal to c then either p(t;k) or p(t;-k) must tend to infinity in the direction k. This is illustrated in Figure 5.a. If no such k exists then D must be non-diagonalizable and $c_\perp$ its only direct eigenvector. In this case, since all manifolds of (7ℓ) intersect the equilibrium set of (7) only once (excluding, of course, that set itself) as depicted in Figure 5.b, and since the solutions on these manifolds are mirror reflections across the origin (from section 2), trajectories on one side of the zero line of (7) must tend to infinity.

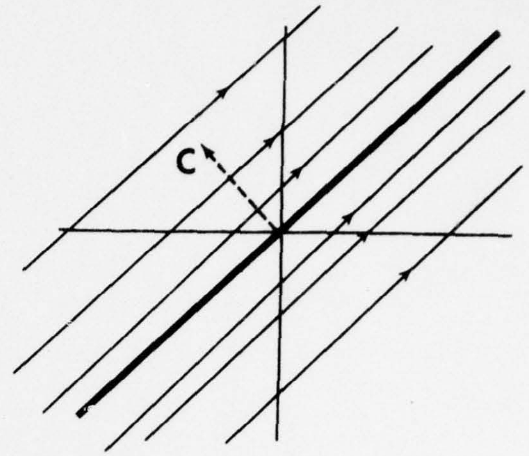


Figure 4. Type (ii): field direction aligned with unique zero line

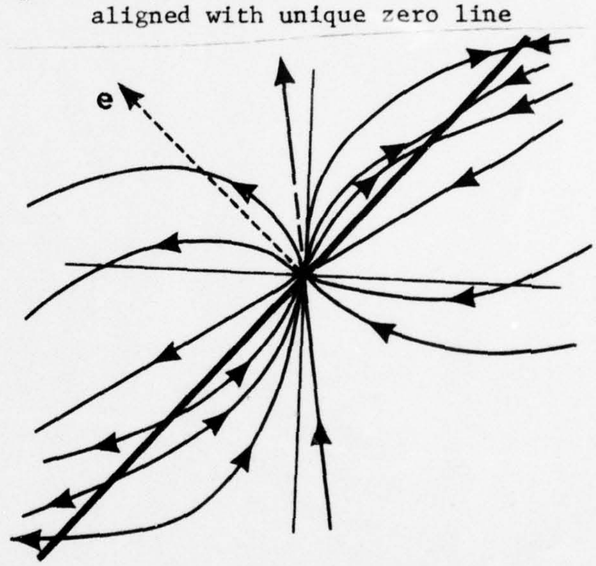


Figure 5.a. Type (ii): Pure Nodal Behavior

By this argument we have excluded all systems of form (7) whose D matrix has real eigenvalues. Thus the following is a statement of the refined necessary conditions: the quadratic differential equation (1) is stable only if its field vanishes on a single line through the origin and its associated linear system (7ℓ) exhibits center or focus (whether stable or unstable) behavior. We will now demonstrate that this condition is also sufficient.

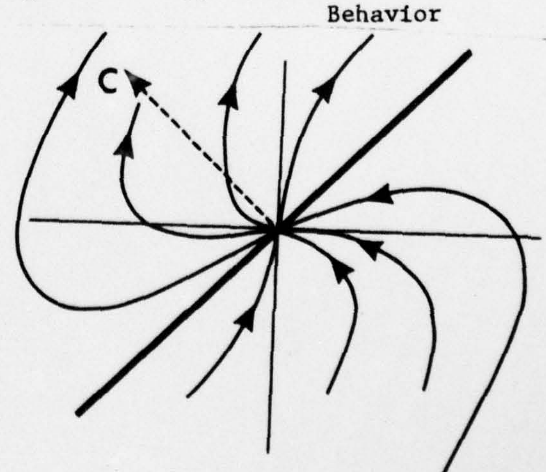


Figure 5.b. Type (ii): Partial Nodal Behavior

c. <u>D with Complex Conjugate Eigenvalues</u>:

If the solution manifolds of (7ℓ) intersect every direction on the plane at a finite point at least once then, recalling the condition from section 2 that every trajectory of (1) must be contained in a half plane we might suspect that (7) is stable when D has complex conjugate eigenvalues. In this case, the containing half plane is defined by the zero line $c_{\perp}$ and the solution to (7) is trapped on a half loop of the spiral or circle defined by the solution of (7ℓ).

This can be argued more formally. Define $L(x_0) \overset{\Delta}{=} \{y \in \mathbb{R}^2 | (t < \infty)(y = e^{Dt}x_0)\}$ - the manifold cut out by a particular solution of (7ℓ) given initial state x_0, and the half plane $P \overset{\Delta}{=} \{y \in R^2 | y^T x_0 > 0\}$. If $p(t;x_0)$ is a solution of (7) then $(\forall t)\ p(t;x_0) \in L(x_0) \cap P$. Since (1) is well behaved, the fact that $L(x_0) \cap P$ may be disconnected is of no concern-we assume that $p(t;x_0)$ lies entirely on the branch, A, of $P \cap L(x_0)$ which is connected to x_0. Clearly A is contained within the closed half-annulus of radius $\|x_0\| \pm \delta$ for some constant δ. Having ruled out the possibility of closed paths we must have (by Bendixson Theory) $p(t;x_0) \longrightarrow \tilde{x}$, $\tilde{x} \in \overline{P \cap L(x_0)}$. Inspecting (7) it is obvious that $\tilde{x}$ lies on the line orthogonal to c. Hence for an arbitrarily small $\varepsilon > 0$ we can always choose $\|x_0\|$ small enough to obtain $p(t;x_0) \in B_\varepsilon (\forall t > t_0)$. The system is stable. Since (1) is homogeneous any stability characteristic must be global. Hence we have the following:

<u>Theorem</u>: The second order quadratic differential equation (1) is globally stable if and only if it is of the form

$$\dot{x} = c^T x D x$$

and the eigenvalues of the matrix D are complex.

<u>*Example*</u>: If $G = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$ and $H = \begin{bmatrix} -.76 & -.15 \\ -.15 & 0 \end{bmatrix}$ then $\dot{x} = c^T x D x$ with $c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 1 \\ -.76 & -.30 \end{bmatrix}$. Since the eigenvalues of D are complex the system is stable.

The system was simulated and results are plotted in Figure 6.a.

<u>*Example*</u>: If $G = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$ and $H = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$ then $\dot{x} = c^T x D x$ with $c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

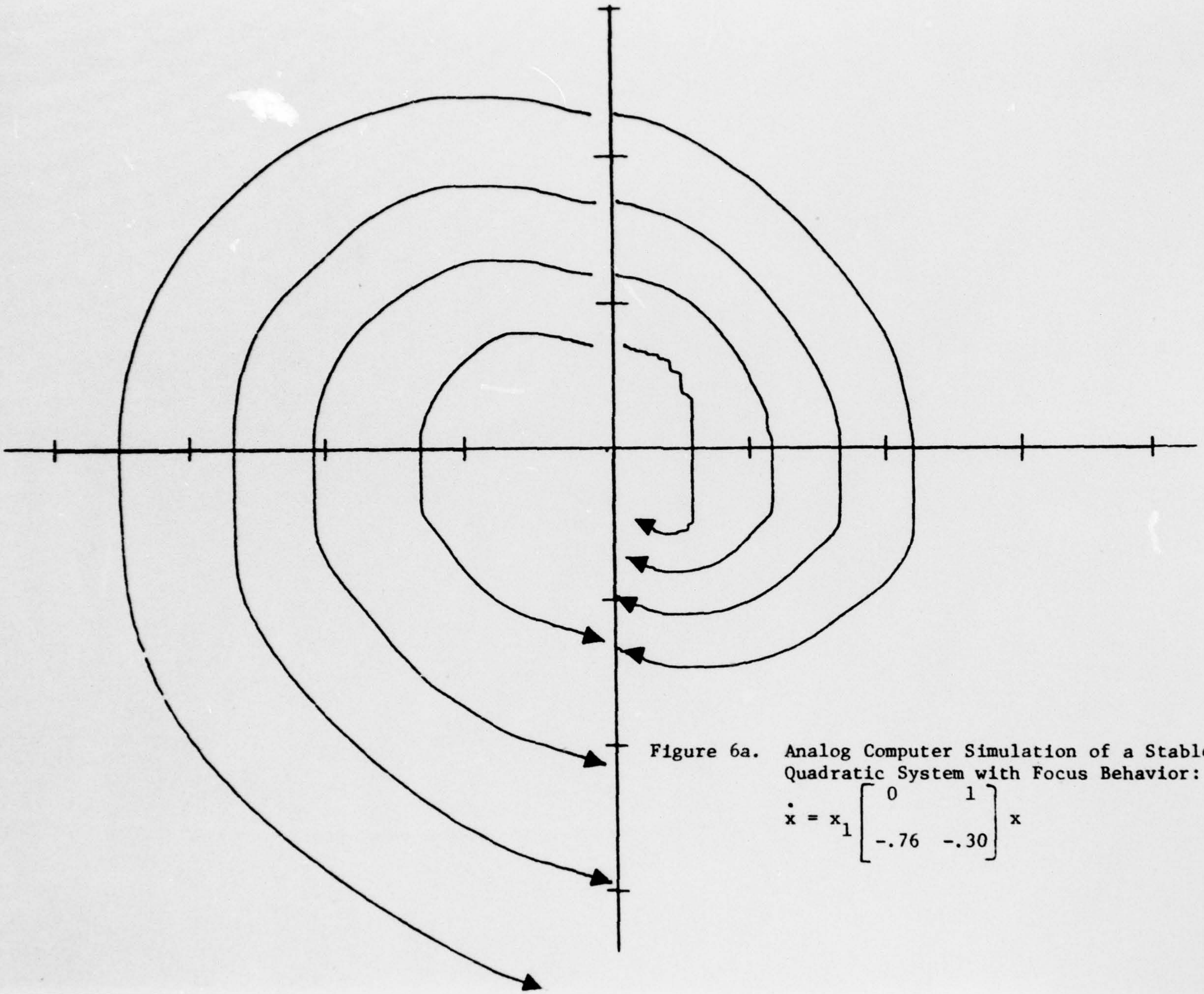


Figure 6a. Analog Computer Simulation of a Stable Quadratic System with Focus Behavior:

$$\dot{x} = x_1 \begin{bmatrix} 0 & 1 \\ -.76 & -.30 \end{bmatrix} x$$

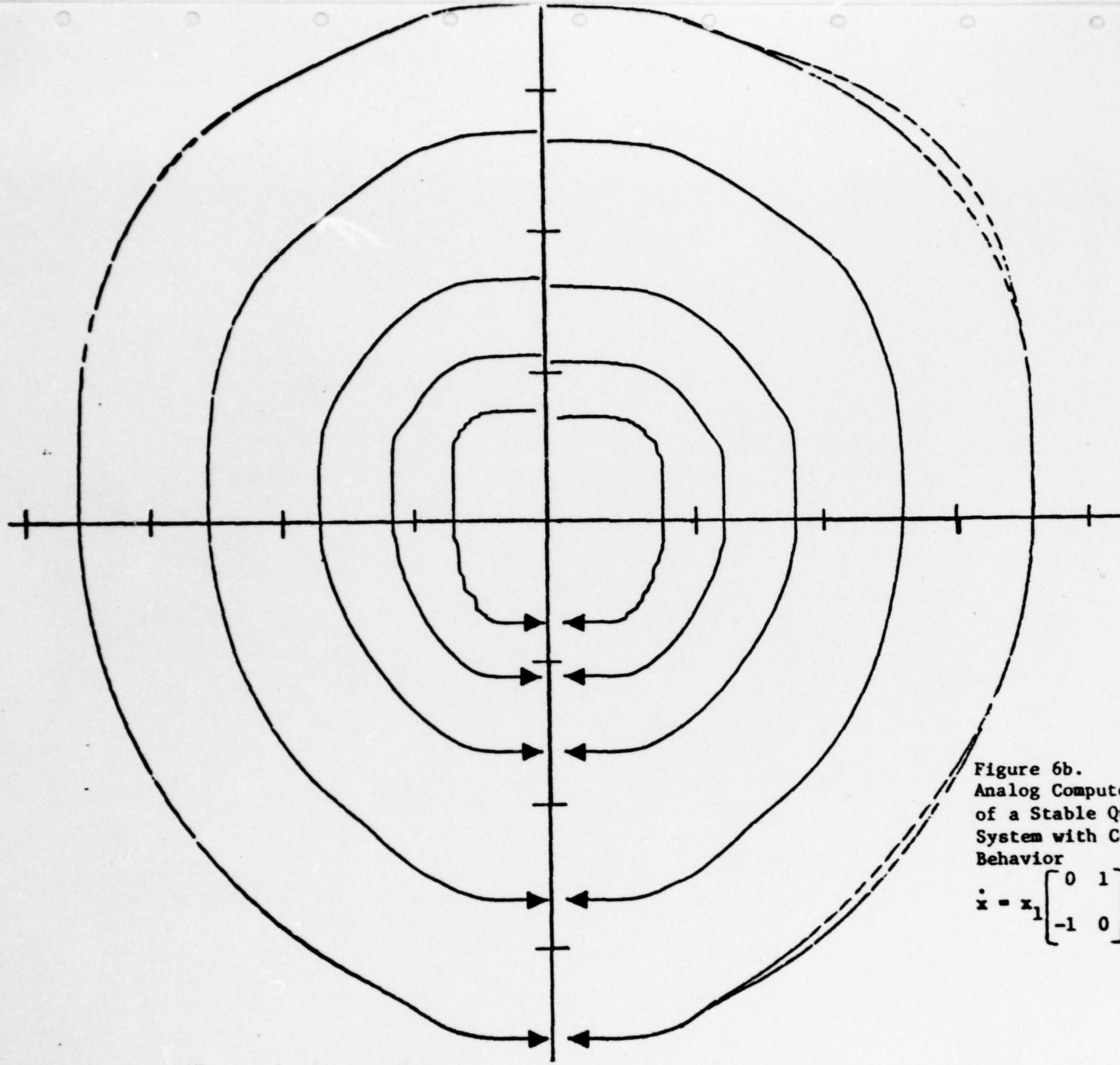


Figure 6b.
Analog Computer Simulation
of a Stable Quadratic
System with Center
Behavior

$$\dot{x} = x_1 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x$$

and $D = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Since the eigenvalues of D are imaginary the system is stable.

The system was simulated and results are plotted in figure 6.b.

5. Global Behavior of Second Order Quadratic Systems:

Results of earlier sections indicate that most quadratic systems are unstable. Hence the derivation of necessary and sufficient conditions for stability cannot address the qualitative behavior of the solutions in most instances of equation (1). To fill this gap, and since the investigation of instability behavior is an intrinsically important component of any stability analysis, this section will be concerned with the classification of the global properties of second order quadratic systems.

a. Polar Coordinate Representation:

For the purposes of this investigation it is most convenient to express equation (1) in polar form. Using the transformation $\rho \triangleq \sqrt{x_1^2 + x_2^2}$, $\Theta = \arctan \frac{x_2}{x_1}$ we may write

$$\dot{\rho} = 1/\rho(x_1\dot{x}_1+x_2\dot{x}_2)$$
$$\dot{\Theta} = 1/\rho^2(x_1\dot{x}_2-x_2\dot{x}_1)$$

where $\dot{x}_1 = x^T G x$ and $\dot{x}_2 = x^T H x$ from (1). If we define $g(v) \triangleq g_2v^2 + g_0v + g_1$ and $h(v) \triangleq h_2v^2 + h_0v + h_1$,* and let $\Theta \in (-\pi/2,\pi/2)$ then this may be re-written as

$$\begin{aligned} \dot{\rho} &= \rho^2 \cos^3 \Theta[g(\tan \Theta) + \tan \Theta\, h(\tan \Theta)] \\ \dot{\Theta} &= \rho \cos^3 \Theta[h(\tan \Theta) - \tan \Theta\, g(\tan \Theta)] \end{aligned} \qquad (9)$$

Recalling from section 2 that the solutions of (1) behave symmetrically across any line through the origin, it is clear that there is no loss of generality in restricting Θ to the given open interval.

* Let $G = \begin{bmatrix} g_1 & 1/2\, g_0 \\ 1/2\, g_0 & g_2 \end{bmatrix}$ and $H = \begin{bmatrix} h_1 & 1/2\, h_0 \\ 1/2\, h_0 & h_2 \end{bmatrix}$

Once again, the stability analysis of (9) requires an identification of its equilibrium states. If $\rho \equiv 0$, then $\dot{\rho} = \dot{\theta} = 0$; the abscissa in the new coordinate system (the origin in (1)) is an equilibrium set. For $\rho \neq 0$, the equilibrium states of (9) are completely determined by θ, which is a direct consequence of homogeneity. Parametrizing the open right half plane by the scalar $v \triangleq \tan\theta = \frac{x_2}{x_1}$, the critical points of (9) are obtained by simultaneously solving the equations

$$
\begin{aligned}
r(v) &\triangleq g(v) + vh(v) = 0 \\
p(v) &\triangleq h(v) - vg(v) = 0
\end{aligned}
\tag{10}
$$

where r or p (or both) are cubic polynomials in v. In the sequel we will assume that p is cubic: i.e. $g_2 \neq 0$ with no loss of generality. (If $g_2 = 0$, a new Cartesian system may be chosen by an orthogonal transformation. This is equivalent to the choice of a different parametrization of θ). Thus $p(v) = 0$ must have either three real roots, two real roots (one with multiplicity 2), or a unique real root.

The following result summarizes the character of solutions of (10), and, hence, the equilibria of (9).

Lemma: v_0 is a real root of (10) if and only if it is a real root of the simultaneous equations $g(v) = 0$, $h(v) = 0$.

Proof: The forward direction is trivial. To prove the converse, assume that v_0 is a real root of (10) (i.e. $r(v_0) = 0 = p(v_0)$), but $g(v_0) \neq 0$. In this case we have $\frac{h(v_0)}{g(v_0)} = v_0 = \frac{-1}{v_0}$: i.e. v_0 must be purely imaginary - contradiction.

By this lemma, (10) must have either (i) no solutions, (ii) one solution, or (iii) two solutions, with the following characteristics:

(i) If equation (10) has no solutions, system (9) has the trivial equilibrium set $\rho = 0$. However, since p is cubic there must be at least one real v_0 such that

$p(v_0) = 0$. For any such roots the relation $\frac{h(v_0)}{g(v_0)} = v_0$ must be true.

(ii) If equation (10) has a single solution v_0, then system (9) has an equilibrium set along the single line $\Theta = \arctan v_0$. If p has other distinct roots, then they cannot be roots of g by the lemma: hence the relation $\frac{h(v)}{g(v)} = v$ must be satisfied at these points. If v_0 is the unique real root of p then the relation $\frac{h(v)}{g(v)} = v$ is never satisfied for $v \in R$. (Unless v_0 has multiplicity 3 in p).

(iii) If equation (10) has two distinct solutions v_0 and v_0' then system (9) has equilibria along two lines $\Theta = \arctan v_0$ and $\Theta = \arctan v_0'$. Since p is cubic and must have a third root, v_0'', which cannot solve (10), the relation $\frac{h(v_0'')}{g(v_0'')} = v_0''$ will hold.

It is clear that these three cases correspond, respectively, to the three types of equation (1) identified in section 3. Using the results of section 4 we may immediately see that the necessary and sufficient condition for stability is given by the second alternative of case (ii). This may be confirmed independently as follows.

Stability Analysis of Polar Form:

Assume case (ii) holds and v_0 is the unique real root of $p(v) = 0$. Then for some $0 < M < \infty$, $-M \leq \frac{r(v)}{p(v)} \leq M$. Hence, from (9), we have $\dot{\rho} = \rho\frac{r(v)}{p(v)}\dot{\Theta} \leq \rho M\dot{\Theta}$. Since $\rho \geq 0$ we may write

$$\frac{d}{dt}\ell n\rho \leq M\frac{d\dot{\Theta}}{dt} \tag{11}$$

Now, factoring $p(v) = (v-v_0)p'(v)$, we must have $p'(v)$ bounded away from zero. If p' is always negative let $\phi \triangleq \Theta - \Theta_0$, if p' is positive, then let $\phi \triangleq \Theta - \Theta_0 - \pi$. Then (assuming the former) we have

$$\frac{d\phi}{dt} = \rho(t)\cos^2(\phi+\Theta_0)p'[\tan(\phi+\Theta_0)]\left[(\tan(\phi+\Theta_0) - v_0)\cos(\phi+\Theta_0)\right]$$

or, multiplying the polynomials in $\tan\Theta$ through by $\cos\Theta$

$$\frac{d\phi}{dt} = [\sin(\phi+\Theta_0) - v_0\cos(\phi+\Theta_0)]\alpha(t,\Theta) \tag{12}$$

$$\phi\epsilon(-\Theta_0-\pi, -\Theta_0+\pi)$$

where $\alpha(t,\Theta)$ is a negative definite function. Since $\phi[\sin(\phi+\Theta_0) - v_0 \cos(\phi+\Theta_0)] \geq 0$ equation (12) is an asymptotically stable scalar system, and $\phi(t) \to 0$ for all initial conditions $\phi_0 \in (-\Theta_0-\pi, -\Theta_0+\pi)$.

Since $\ln \rho$ is a positive function, the stability of (12), and the majorization in (11) forces the boundedness of $\ln \rho$. Of course $\ln \rho$ is bounded if and only if ρ is bounded. Then if $\frac{h(v)}{g(v)} = v$ does not hold for any v, equation (1) is globally stable.

Conversely, it may be similarly demonstrated, independent of the analysis in section 4 that all remaining cases of equation (9) are unstable. Recall that in each remaining case there exists at least one real v_0 such that $\frac{h(v_0)}{g(v_0)} = v_0$ and $r(v_0) \neq 0$.* Then for initial conditions (ρ,Θ_0) when Θ_0 = arc tan v_0, system (9) may be re-written as:

$$\begin{aligned} \dot{\rho} &= \rho^2 \cos^3 \Theta_0 \, r\,(\tan \Theta_0) \\ \dot{\Theta} &= 0 \end{aligned} \qquad (13)$$

which is a scalar quadratic system. Since $\rho \geq 0$, equation (8) has bounded solutions if and only if $\cos^3 \Theta_0 \, r\,(\tan \Theta_0) < 0$. However if this is the case then $\cos^3 (\Theta_0 + \pi)$ $r[\tan(\Theta_0 + \pi)] > 0$ and $p(\tan (\Theta_0 + \pi)) = 0$. Hence initial condition $(\rho, \Theta_0 + \pi)$ also yields equation (13) and has the solution

$$\rho(t) = \frac{1}{-(2 \cos(\Theta_0 + \pi) \tan \Theta_0)t - \rho_0}$$

which escapes to infinity in finite time.

In summary, we have re-stated the necessary and sufficient conditions for stability of equation (1) in polar form: system (9) is stable if and only if there is no $v_0 \in \mathbb{R}$ such that $\frac{h(v_0)}{g(v_0)} = v_0$.

<u>Relation of Polar to Cartesian System:</u>

The crucial equation $\frac{h(v)}{g(v)} = v$ is intimately related to the quadratic operator in (1): the existence of a real v_0 to solve that equality is equivalent to the

* Excluding the case where the equality holds at v_0, and v_0 is simultaneously a triple root of p.

existence of a non-zero fixed direction of f.*

Suppose $f(x_0) = \lambda x_0 (\lambda \neq 0,\ x_0 \neq 0)$. We have $\dfrac{x_0^T M x_0}{x_0^T G x_0} = \dfrac{\lambda x_{02}}{\lambda x_{01}} = \dfrac{x_{02}}{x_{01}}$.

But $\dfrac{x^T M x}{x^T G x} = \dfrac{h(\frac{x_2}{x_1})}{g(\frac{x_2}{x_1})}$. Hence letting $v_0 \triangleq \dfrac{x_{02}}{x_{01}}$ we have $\dfrac{h(v_0)}{g(v_0)} = v_0$.

The converse, while strongly suggested by the condition $\dot\rho \neq 0$, $\dot\theta = 0$ is immediate if we recall the identity

$$\left.\frac{\dot x_2}{\dot x_1}\right|_{x(0) = x_0} \equiv \frac{dx_2}{dx_1}$$

where $x_2(x_1)$ is a local parametrization of the phase curve $\{p(t;x_0) | t > 0\}$ in $\mathbb{R}^2$. Namely, the ratio of the time derivatives specifies the slope of the tangent line of the manifold on which a solution lies. Then the condition $\dfrac{h(v_0)}{g(v_0)} = v_0$ is equivalent to the condition $\dfrac{dx_2}{dx_1} = \dfrac{x_2}{x_1}$, and trajectory $p\left(t;\begin{bmatrix}\alpha\\ \alpha v_0\end{bmatrix}\right)$ lies on the line $x_2 = v_0 x_1$. Hence $f\left(\begin{bmatrix}\alpha\\ \alpha v_0\end{bmatrix}\right) = \lambda\left(\begin{bmatrix}\alpha\\ \alpha v_0\end{bmatrix}\right)$ and since $r(v_0) \neq 0$ implies $\dot\rho \neq 0$, we must have $\lambda \neq 0$.

<u>Necessary and Sufficient Conditions for Stability</u>:

In summary we have shown that the following statements are equivalent

<u>Stability</u>:

(i) Equation (1) is globally stable.

(ii) $f(x) = c^T x Dx$ and D has pure complex eigenvalues.

(iii) The equation $\dfrac{h(v)}{g(v)} = v$ has no real solutions.

(iv) The quadratic operator f has no non-zero fixed directions.**

and are mutually exclusive of the set of equivalent statements

<u>Instability</u>:

(i) Equation (1) has a finite escape trajectory.**

(ii) f is type (i) or D has real eigenvalues.

(iii) There exists a $v_0 \in R$ such that $\dfrac{h(v_0)}{g(v_0)} = v_0$.

(iv) The quadratic operator f has a non-zero fixed direction.**

* Excluding the case where the equality holds at v_0, and v_0 is simultaneously a triple root of p. (see next footnote)

** Excluding the situation where $f(x) = c^T x Dx$, D has a double real eigenvalue, and the only eigenvector of D is on the orthogonal complement of c. This is unstable, as shown in section 4. It is equivalent to the condition that v_0 is a triple root of p in case (ii).

b. Classification of Unstable Quadratic Systems:

Having reiterated earlier results using the polar coordinate transformation, we will now classify the various types of instability behavior solutions to system (1) may evince on the basis of their associated phase curves.

It has already been established that the solution manifolds of type (ii) and (iii) systems given by equation (7) (i.e. where $f(x) = c^T x D x$) are identical to those which are associated with the linear system (7ℓ).

$$\dot{z} = \beta D z \tag{7ℓ}$$

It is well-known that system (7ℓ) may possess nodal, focus, or center characteristics. In the case of focus or center behavior of (7ℓ) we have demonstrated that (1) is stable. Otherwise the trajectories of (7) follow half the nodal manifolds (7ℓ) to the origin, and the other half out to infinity, with trajectories starting on an eigenvector of D escaping to infinity in finite time, [Figures 3-6].

On the other hand, we have said very little about the trajectories of systems of type (i). Again, it is more convenient to examine the solutions of (9) than to attempt an immediate analysis of the Cartesian system (1). The tangent slope of phase curves in the (ρ,Θ) plane is given by

$$q(v) \stackrel{\Delta}{=} \frac{d\rho}{d\Theta} = \rho \, \frac{r(\tan\Theta)}{p(\tan\Theta)} = \rho \, \frac{g(v) + vh(v)}{h(v) - vg(v)} \tag{12}$$

The poles of q correspond to the values of v such that $\frac{h(v)}{g(v)} = v$ as established before. The significant characteristics of the manifolds of (1) are determined quite easily by a glance at the sign of q in the neighborhood of a pole. If v_0 is a single root of $p(v) = 0$ then $q(v_0 + \varepsilon)q(v_0 - \varepsilon) < 0$. If $q(v_0 + \varepsilon) > 0$ then solution manifolds on the (ρ,Θ) form a negative cusp around the line $\Theta = \arctan v_0$ as shown in figure 7.a. If $q(v_0 + \varepsilon) < 0$ then solution manifolds on the (ρ,Θ) plane form a positive cusp around the line $\Theta = \arctan v_0$, as shown in figure 7.b. If v_0 is a double root of $p(v) = 0$ then $q(v_0 + \varepsilon)q(v_0 - \varepsilon) > 0$ and the solution manifolds will have positive (or negative) slope on both sides of the line Θ = arc tan v_0 as shown in figure 7.c.

<u>Loops and Flows</u>:

In the light of the above observations, a rough translation of figures 8 into the Cartesian plane convinces us that the prototypic behavior of system (1) type (i) falls into two distinct classes. The trajectories must be separated by at least 1 and up to 3 separatrices: the fixed directions of the map f as determined by the zeros of p. If f has a unique fixed direction, then the solutions of (1) must correspond to either figure 7.a or 7.b. In the (x_1, x_2) plane these prototypes are given by figure 8.a or 8.b which we will call the "loop" or "flow" prototypes respectively. If f has two fixed directions, then p must have one double root. In this case, one fixed direction will have either the loop, or the flow characteristic, and the other will have the special feature depicted in 7.c. The solutions in the (x_1, x_2) plane retain either a flow or a loop character as shown in figure 8.c (flow dominant) and 8.d (loop dominant). Finally, if f has three fixed directions, p must have three roots and each root must evince either pure loop or flow characteristics.

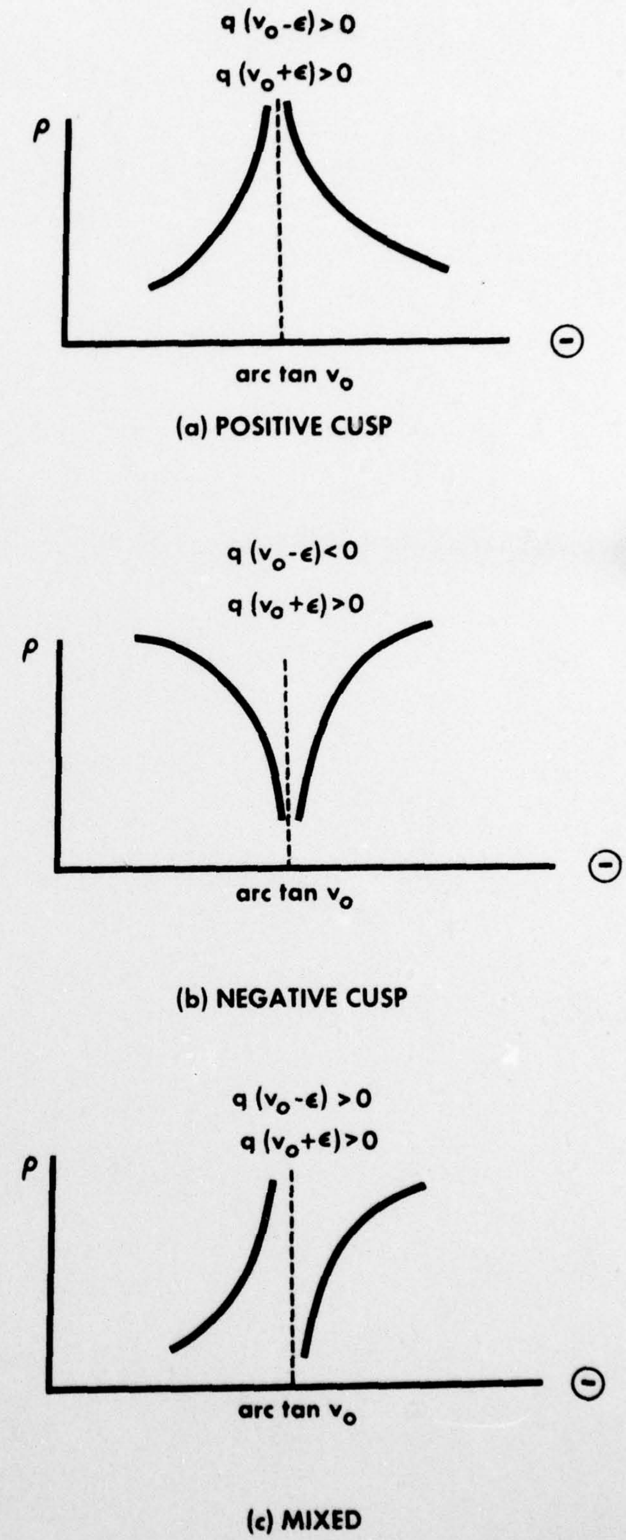


Figure 7. Characteristic Manifolds on the Polar Plane

Illustrative examples of these cases are now given:

Example 5.1.a. <u>Flow with one fixed direction</u>:

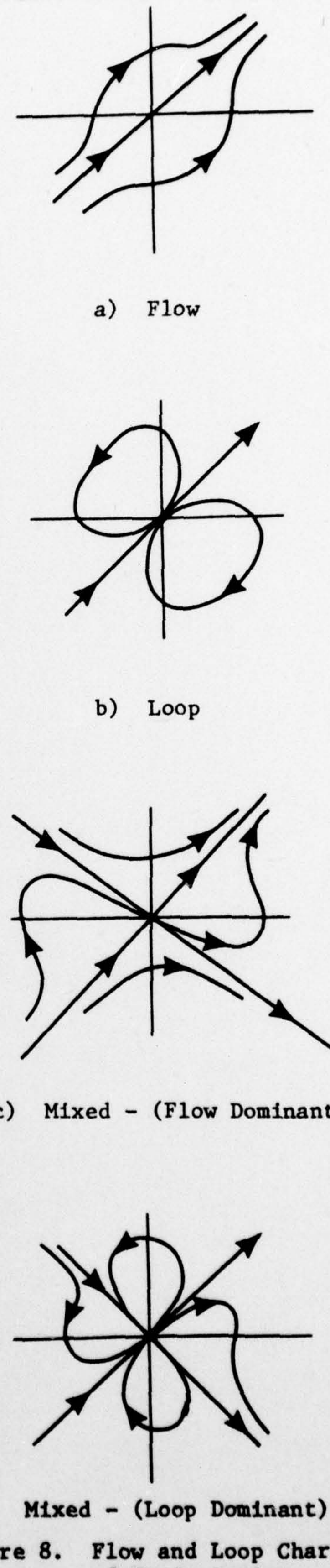
$$\text{Let } G = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \text{ Then}$$

$g(v) = v^2, h(v) = 1$ and $r(v) = v(v + 1)$, $p(v) = 1 - v^3$. The system is type (i) since g and h have no common factors. The cubic polynomial p has a single root at $v = 1$, hence the direction $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is invariant under f. The ratio $g(v) = \dfrac{r(v)}{p(v)}$ is positive in the left-hand neighborhood of 1. Hence the tangent lines $\frac{d\rho}{d\Theta}$ are given by figure 1.b. The actual system was simulated on the digital computer and results are plotted in figure 9.a.

Example 5.1.b. <u>Loop with one fixed direction</u>:

$$\text{Let } G = \begin{bmatrix} 1 & 1/2 \\ 1/2 & -1 \end{bmatrix} \text{ and } H = \begin{bmatrix} -1 & 1/2 \\ 1/2 & 1 \end{bmatrix}.$$

Then $g(v) = -(v^2-v-1)$, $h(v) = v^2+v-1)$ and $r(v) - v^3 + 1$, $p(v) = v^3 - 1$. Again the system is type (i), and p has a unique root $v = 1$. The vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is again fixed under f. Now, however, the ratio $q(v)$ is negative in the left-hand neighborhood of 1. Hence $\frac{d\rho}{d\Theta}$ is given by 1.a. The actual system was simulated on the digital computer and results are plotted in figure 9.b.



a) Flow

b) Loop

c) Mixed - (Flow Dominant)

d) Mixed - (Loop Dominant)

Figure 8. Flow and Loop Characteristics of Trajectories

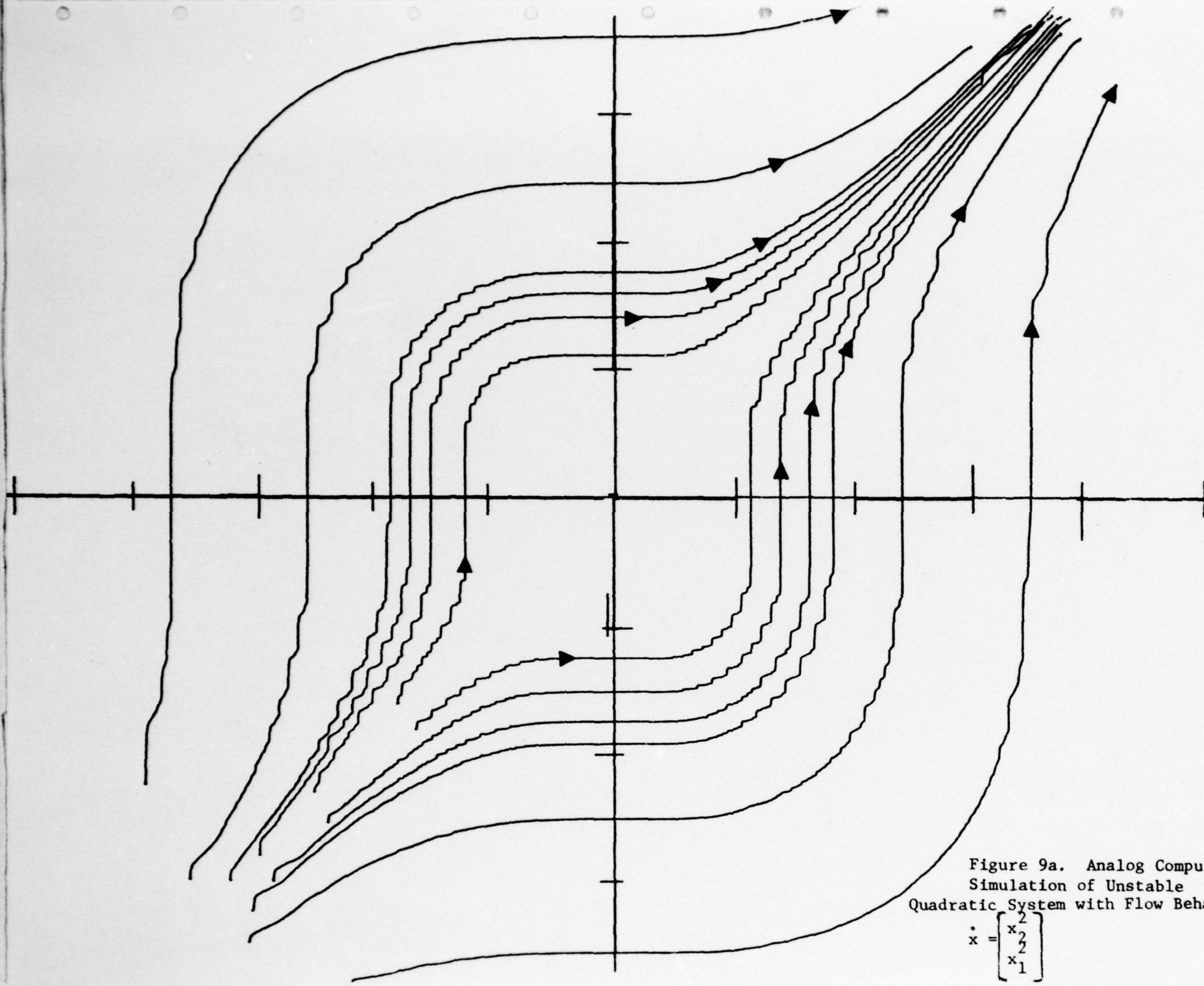


Figure 9a. Analog Computer Simulation of Unstable Quadratic System with Flow Behavior.

$$\dot{x} = \begin{bmatrix} x_2^2 \\ x_1^2 \end{bmatrix}$$

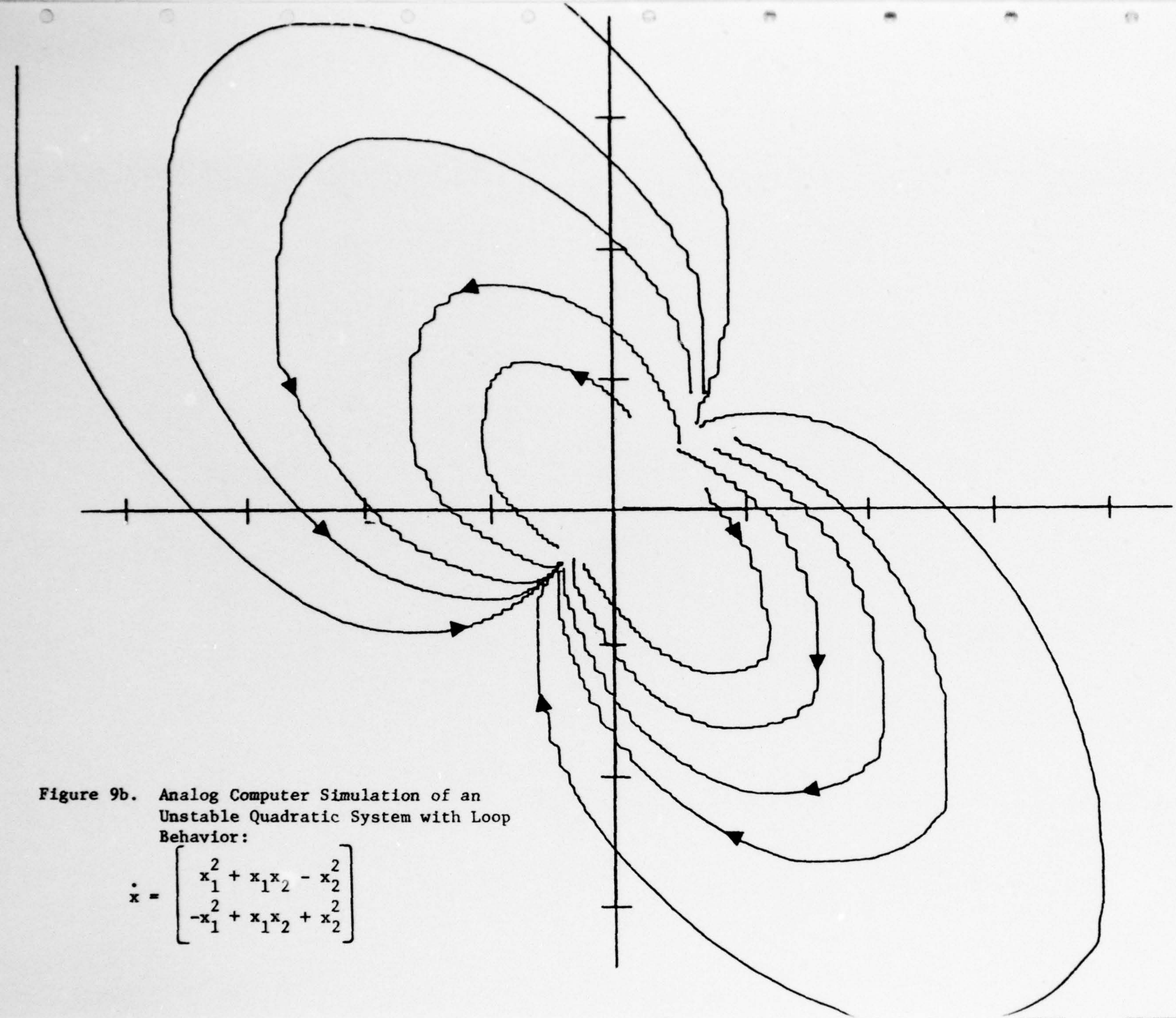


Figure 9b. Analog Computer Simulation of an Unstable Quadratic System with Loop Behavior:

$$\dot{x} = \begin{bmatrix} x_1^2 + x_1x_2 - x_2^2 \\ -x_1^2 + x_1x_2 + x_2^2 \end{bmatrix}$$

c. The Stability of Off-Origin Equilibria:

In the preceding discussion a polar coordinate transformation elucidated the analysis of the global behavior of equation (1) with respect to the origin. To complete the investigation of global behavior an account of the stability properties of other equilibria are presented here.

Systems of type (i), with no other equilibria than the origin have been treated in the previous subsection. For the two remaining types, recall from section 4 that equation (1) may be expressed using equation (7)

$$\dot{x} = c^T x D x$$

where D is either of full rank or singular. In the former case the system has a unique line of equilibrium i.e. $x = c_\perp$. In the latter case the system has two lines of equilibria which satisfy $x = c_\perp$ and $Dx = 0$ [a special case occurs when these two lines coincide].

Systems of Type (iii):

If D is singular, it was shown in section 4.a that equation (7) can be rewritten as equation (8):

$$\dot{x} = k x^T G x$$

This is type (iii) if and only if G is indefinite and of full rank. As explained in that discussion solutions lie on parallel lines in the direction k. These, in turn, correspond to the solution manifolds of a singular linear system whose unique non-zero eigenvector is k. Superimposed on this field is the equilibrium set specified by the two distinct vectors e_1 and e_2 for which $x^T G x\big|_{x=e_i} = 0$ $(i = 1,2)$ is satisfied. In appendix 1 we present a formal analysis of the perturbed motion around the equilibria λe_1 and λe_2, $\lambda \in R$.

If $k \neq e_i$ $(i = 1,2)$ then either positive or negative open half line (i.e. excluding the origin) contains stable equilibria and its opposite closed half line (i.e. including the origin) contains unstable equilibria. Each stable half line is a locally attractive invariant set. This situation is portrayed in figure 3.a (section 4).

If $k = e_1$ then the direction of the field is itself a line of equilibria, all of which must be unstable. In this case, the remaining line of equilibria has an open half line of stable and a closed half line of unstable points. The unique stable half line is a locally attractive invariant set. This situation is portrayed in figure 3.b (section 4).

Systems of Type (ii):

As established in section 3 if D is nonsingular (or singular and possesses a zero eigenvector in the direction orthogonal to c) then the system is type (ii). Again the solutions lie on manifolds determined by the associated linear system

$$\dot{z} = Dz$$

interrupted by the zero line. In appendix 1 it is formally shown that in all cases there must be a half line of stable equilibria (excluding again the case where the only eigenvector of D is orthogonal to c). The half line is closed if and only if D has purely complex eigenvalues. This line is a locally attractive invariant set. Two representative systems are depicted in figures 5 (section 4).

Appendix I: The Stability of Off-Origin Equilibria

Recall that system (1), $\dot{x} = f(x)$, may be re-written as equation (7), $\dot{x} = c^T x D x$, if there are any off-origin equilibria. Around an arbitrary equilibrium state, $\lambda e (e^T c = 0$ and $\lambda \in \mathbb{R})$, the perturbed equation of motion $\dot{z} = f(e + z) - f(e)$ may be simplified and written as:

$$\dot{z} = c^T z D \lambda e + c^T z D z$$

Define the matrix $R = [De \vdots e]$ and the change of basis $y = R^{-1} z$.*
Then $\dot{y} = \lambda R^{-1} D e c^T R y + R^{-1} D R y c^T R y$. Since $c^T R = [c^T De, 0]$ and $R^{-1} De = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ we may re-write the last equation as

$$\dot{y} = \lambda y_1 c^T De \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + R^{-1} D R y \right) \tag{A.1}$$

Fact: System (A.1) is locally stable if and only if $\lambda c^T De < 0$ in which case the line $y_1 = 0$ is an attractive set.

Proof: If $\lambda c^T De > 0$ let $v = y_1^2$. Then $\dot{v} = \lambda c^T De y_1^2 + y_1 0(y^2)$. Since $v > 0$ in the region $\dot{v} > 0$ for small enough values of $\|y\|$, the system is unstable by Chetaev's theorem.

If $\lambda c^T De < 0$, let $v = 1/2 y^T y$. Then $\dot{v} = \lambda c^T De y_1^2 + y_1 0(y^2)$ which is negative semi-definite in a small enough neighborhood of the origin, Ω. Since $\dot{v}$ is always on $\{x_1 \neq 0\} \cap \Omega$ we know $\{x_1 = 0\} \cap \Omega$ is attractive. Since the equation of perturbed motion about any point in $\{x_1 = 0\}$ is of the form (A.1), the same argument works on the entire line.

If the system is type (iii) then using equation (8) we know

$$\dot{x} = k x^T G x = k x^T b d^T x \tag{8}$$

Thus the matrix D in (7) may be either kb^T or kd^T with $c = d$ or $c = b$ respectively. If $k \neq d_1$ and $k \neq b_1$ then $\lambda c^T De < 0$ for either $\lambda < 0$ or $\lambda > 0$ using either form of

---

* Of course this is not possible if $De = \lambda e$. In this case, either choose the second zero line (if the system is type (iii)) and perform the identical analysis, or if the system is type (ii) then there is no attractive half line. (see second footnote on p. 16).

D. Thus there are two attractive open half lines. If $k_1 = d$ or $k_1 = b$ then $c^T De = 0$ for one choice of D. Hence there is only one open attractive half line.*

If the system is type (ii) then the matrix D in equation (7) is unique and either nonsingular or equal to cc^T (see section 4). If $c^T De \neq 0$ then there must be an attractive half line (open or closed depending upon the stability of the origin of course). Notice $c^T De = 0$ only if e is the unique eigenvector of D^* (see footnote on previous page).

---

* Of course this is not possible if $De = \lambda e$. In this case, either choose the second zero line (if the system is type (iii)) and perform the identical analysis, or if the system is type (ii) then there is no attractive half line. (see second footnote on p. 16).

Acknowledgment

This project was supported in part by the Office of Naval Research under Contract N00014-67-A-0097-0020, NR 375-131.

References

[1] Frayman, Morris, Quadratic Differential Systems: A Study in Nonlinear Systems Theory (1974), Doctoral Dissertation, University of Maryland.

[2] Hahn, Wolfgang, Stability of Motion, Springer Verlag, Berlin 1967.

[3] LaSalle, J. F. and Lefschetz, S., Stability by Liapunov's Direct Method with Applications, Academic Press, New York, 1961.

[4] Lyaghina, L. S., "The integral curves of the equation $y^1 = \frac{ax^2+bxy+cy^2}{dx^2+exy+fy^2}$," Uspekhi Mat. Nauk, VI, 2(42) 171-83, 1951 (Russian).

[5] Lüders, G. and Narendra, K. S., "Lyapunov Functions for Quadratic Differential Equations with Applications to Adaptive Control," IEEE Trans. on Auto. Control, Dec. 1972.